

Designing a realistic mathematics learning trajectory to promote fraction concept construction in elementary school

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Abstract

Purpose: This study evaluates the effectiveness of a Hypothetical Learning Trajectory grounded in Realistic Mathematics Education to resolve primary students' persistent conceptual bottlenecks in fraction equivalency, operations, and word problem solving.

Method: Utilizing a two-cycle classroom action research model, the study investigated a primary school student cohort. Data were collected using validated diagnostic assessments, classroom observation protocols, student work artifact analysis, and semi-structured interview transcripts.

Findings: Quantitative assessments demonstrated progressive gains across evaluation phases, with average test scores and classical learning mastery showing substantial improvement and successfully exceeding the target success threshold. Qualitative artifact and interview analyses confirmed a pronounced cognitive shift, as students transitioned from misapplying mechanical algorithms to demonstrating model-based partitioning, visual representation, and verbal justifications.

Practical implications: The validated learning trajectory provides elementary educators with an empirically supported pedagogical framework, featuring actionable task sequences, strategic linguistic scaffolding tools, and problem-posing activities that bridge informal situational reasoning with formal symbolic mathematics.

Originality/value: This research advances local instruction theory by demonstrating how student-led problem posing and strategic scaffolding within a realistic mathematics environment systematically facilitate both horizontal and vertical mathematization.

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INTRODUCTION

Mathematics education at the elementary level forms the essential foundation for students' logical reasoning, problem-solving skills, and abstract mathematical thinking. Within this foundational landscape, learning trajectories have emerged as pivotal frameworks for mapping out the progressive steps through which students construct mathematical concepts across various domains (Andrews-Larson et al., 2017; Castro et al., 2026; Clements et al., 2004; Ellis et al., 2016; Popa, 2023). Educational design research consistently demonstrates that structuring learning pathways allows educators to anticipate student thinking and foster deeper conceptual understanding rather than procedural memorization (Bakker & van Eerde, 2015; Swidan, 2020). Furthermore, integrating realistic contexts into these pathways bridges the gap between informal student experiences and formal mathematical structures, facilitating meaningful learning trajectories in diverse topics such as geometry, algebra, and calculus (Cárcamo Bahamonde et al., 2017; Gravemeijer & Doorman, 1999).

Understanding fractions and fraction operations is a critical milestone in primary education, serving as a gateway to higher-level mathematical literacy and real-world problem-solving capabilities (Crawford, 2022; Hwang & Ham, 2021). Grounding fraction instruction in authentic, reality-based situations enables young learners to connect abstract symbolic operations with tangible experiences, thereby enhancing their overall mathematical reasoning (Große, 2014; Kramarski et al., 2002; Leiss et al., 2019). When students engage with meaningful context-based tasks, they develop metacognitive and problem-posing abilities that extend far beyond the immediate classroom setting (Bonotto, 2013; Hartmann et al., 2021). Consequently, establishing an effective learning trajectory for fraction concepts carries profound educational value, ensuring that primary school students build a resilient foundation for future academic success.

Despite the recognized importance of context-based mathematics education, conventional elementary instruction frequently reduces fraction learning to

mechanical rules and algorithmic drills without conceptual grounding (Depaepe et al., 2010; Neef et al., 2003). While substantial research has explored learning trajectories in secondary and tertiary mathematics—such as linear algebra, trigonometric equations, and advanced calculus—there remains a notable lack of empirically validated local instruction theories tailored specifically to primary students' fraction concept construction (Andrews-Larson et al., 2017; Castro et al., 2026; Gravemeijer, 2004; Swidan, 2020). Furthermore, while preservice and novice teachers are increasingly tasked with designing rich mathematical problems, studies show they often struggle to craft and implement tasks that effectively guide elementary students through a coherent learning trajectory (Isik & Kar, 2012; Kohar et al., 2019; Şengül & Katranci, 2015; Sevinc & Lesh, 2021). A clear gap exists in designing a systematic, classroom-tested learning trajectory that seamlessly translates realistic context problems into formal fraction understanding for primary learners.

To address this gap, this study introduces a novelty by designing and evaluating a Realistic Mathematics Education (RME)-based hypothetical learning trajectory (HLT) tailored specifically to the conceptual construction of fractions in primary school classrooms (Gravemeijer, 2004; Gravemeijer & Doorman, 1999). Unlike existing studies that predominantly focus on teacher professional development or preservice teacher task design, this research directly examines the interaction between students' emergent thinking and the iterative refinement of a learning trajectory in a live elementary setting (Niess & Gillow-Wiles, 2014; Rees, 2009; Sevinc & Lesh, 2018). Furthermore, this study utilizes a design-based research methodology to trace how students transition from informal, real-world representations to formal mathematical operations through carefully scaffolded tasks (Bakker & van Eerde, 2015; Prediger & Krägeloh, 2015). By explicitly bridging local instruction theory with classroom practice, this research provides a fresh empirical perspective on fraction learning pathways.

The primary objective of this study is to design, test, and refine a realistic mathematics learning trajectory that supports primary school students in constructing a deep conceptual understanding of fractions and fraction operations. Specifically, this research seeks to investigate how elementary students progress from informal contextual strategies to formal mathematical models when engaging with the proposed HLT (Gravemeijer, 2004; Gravemeijer & Doorman, 1999). Ultimately, the study aims to establish a validated local instruction theory that serves as a practical and theoretical model for elementary fraction instruction (Bakker & van Eerde, 2015).

Current educational assessments highlight persistent struggles among elementary students in applying mathematical concepts to real-world context problems and word problem scenarios (Depaepe et al., 2010; Hwang & Ham, 2021). The urgency of this issue is further amplified by modern curriculum demands that prioritize mathematical literacy, authentic problem solving, and language-responsive instruction in primary classrooms (Leiss et al., 2019; Prediger & Krägeloh, 2015). Without an empirically grounded learning trajectory, teachers are left with fragmented instructional tools that fail to address the root causes of student misconceptions in fraction operations (Gravemeijer, 2004; Neef et al., 2003). Designing an accessible and effective learning pathway is therefore an urgent priority for improving classroom practice and equity in primary mathematics education (Crawford, 2022).

This research contributes both theoretically and practically to the field of mathematics education by delivering a fully tested and refined learning trajectory for fraction concept construction in primary schools (Bakker & van Eerde, 2015; Gravemeijer, 2004). Theoretically, it enriches the literature on Realistic Mathematics Education by providing detailed empirical insights into students' cognitive transitions from informal context-based reasoning to formal mathematical models (Gravemeijer & Doorman, 1999; Große, 2014). Practically, the resulting learning trajectory offers elementary teachers, task designers, and teacher educators an actionable framework

and exemplar tasks to foster authentic problem-solving and mathematical literacy in young learners (Bonotto, 2013; Kohar et al., 2019; Sevinc & Lesh, 2018).

METHOD

2.1 Research Design & Setting

This study employed a Classroom Action Research (CAR) design structured around the cyclical model proposed by Kemmis and McTaggart, which encompasses four distinct phases: planning, acting, observing, and reflecting. In accordance with design research principles in mathematics education, this action research framework was systematically integrated with the design, implementation, and iterative refinement of a Hypothetical Learning Trajectory (HLT) (Bakker & van Eerde, 2015; Gravemeijer, 2004). The classroom action research design provided a flexible yet rigorous empirical environment to examine how primary school students progress from informal strategies to formal mathematical models within a live classroom setting (Gravemeijer & Doorman, 1999; Prediger & Krägeloh, 2015). By utilizing iterative reflection cycles, the researcher continuously adjusted instructional interventions to align with students' emergent mental processes and conceptual difficulties during fraction learning (Depaepe et al., 2010; Große, 2014).

The research was conducted at SDIT Raudhaturrahmah Pekanbaru, focusing on a sixth-grade elementary classroom environment. The study subjects consisted of primary school students engaged in learning fraction concepts and operations, alongside their class teacher who acted as a collaborative instructor during the implementation. This specific setting was chosen due to observed challenges in students' ability to solve context-based word problems and authentic mathematical tasks, which often stemmed from traditional, algorithm-heavy instruction (Kramarski et al., 2002; Neef et al., 2003). Furthermore, the educational setting reflected typical primary school classroom dynamics in Indonesia, where transitioning students toward high-level mathematical literacy and PISA-like problem-solving remains an urgent pedagogical target (Hwang & Ham, 2021; Kohar et al., 2019).

To ensure high standards of educational research, the setting was organized to support collaborative small-group work, open discussions, and guided classroom reflection (Nuthall, 2004; Santrock, 2021). The physical and social layout of the classroom was structured to facilitate peer interactions and problem-posing activities, allowing students to freely express their mathematical reasoning (Bonotto, 2013; Şengül & Katranci, 2015). The research team established ethical protocols, obtaining institutional permission and informed consent while maintaining natural classroom conditions throughout the intervention. This ensured that the classroom action research

accurately captured the authentic developmental trajectory of primary learners as they constructed fraction concepts (Clements et al., 2004; Crawford, 2022).

2.2 Instructional Intervention (RME Approach)

The instructional intervention was designed using the principles of Realistic Mathematics Education (RME), focusing on the creation and execution of context-based student worksheets across two distinct action cycles (Gravemeijer, 2004; Gravemeijer & Doorman, 1999). In Cycle I, the intervention focused on introducing realistic context problems—such as fair sharing of food and measuring real-world objects—to activate students' informal prior knowledge regarding fractions (Bonotto, 2013; Leiss et al., 2019). Students worked in small heterogeneous groups to explore fraction concepts, using concrete artifacts and visual representations like area models and fraction bars (Clements et al., 2004; Hartmann et al., 2021). The teacher acted as a facilitator, encouraging students to develop their own informal strategies, pose related mathematical questions, and share their initial solutions through guided classroom presentations (Isik & Kar, 2012; Şengül & Katranci, 2015).

Cycle II was explicitly constructed based on the retrospective analysis and reflections gathered from Cycle I, functioning as an updated and refined learning trajectory (Bakker & van Eerde, 2015; Gravemeijer, 2004). The student worksheets in Cycle II shifted toward bridging informal contextual strategies into formal mathematical representations and operations (Große, 2014; Prediger & Krägeloh, 2015). During this cycle, instructional tasks placed greater emphasis on horizontal and vertical mathematization, prompting students to transition from physical or visual models to symbolic algorithms for fraction addition, subtraction, and multiplication (Ellis et al., 2016; Swidan, 2020). Group discussions were structured to foster metacognitive self-regulation, prompting students to explain, defend, and compare their mathematical models during whole-class presentations (Kramarski et al., 2002; Santrock, 2021).

The design of the intervention placed preservice and in-service teacher expertise at the core of classroom implementation, ensuring that the RME-based tasks were rich, contextually realistic, and pedagogically sound (Chan, 2013; Lee, 2013; Sevinc & Lesh, 2018, 2021). The learning trajectory explicitly integrated strategic scaffolding to assist low-achieving students in processing language-heavy context problems without restricting the conceptual richness of the mathematical tasks (Depaepe et al., 2010; Leiss et al., 2019; Prediger & Krägeloh, 2015). Furthermore, the intervention challenged students to move beyond passive task completion by incorporating problem-posing opportunities where learners constructed their own realistic story problems based on given fractional situations (Hartmann et al., 2021; Stoyanova & Ellerton, 1996). This structured two-cycle RME intervention ensured a gradual, meaningful transition from

realistic experiences to formal fraction mastery (Gravemeijer & Doorman, 1999; Sullivan et al., 2015).

2.3 Data Collection & Analysis

Data collection utilized a comprehensive set of qualitative and quantitative instruments designed to capture both the classroom learning process and individual student learning outcomes (Bakker & van Eerde, 2015; Nyman, 2017). Observation sheets were systematically deployed during every lesson cycle to evaluate both teacher instructional fidelity and student active engagement, group interactions, and mathematical discourse (Niss et al., 2017; Nuthall, 2004). Additionally, diagnostic tests and daily performance assessments featuring authentic context-based problems were administered at the end of each cycle to measure students' conceptual understanding, procedural fluency, and problem-solving growth in fraction operations (Hwang & Ham, 2021; Kramarski et al., 2002). Artifacts of student work, including written solution strategies and student-generated problem-posing sheets, were collected as primary evidence of their cognitive progression along the learning trajectory (Bonotto, 2013; Hartmann et al., 2021).

Data analysis was conducted using a mixed analytical approach that combined qualitative retrospective analysis with descriptive quantitative statistics (Bakker & van Eerde, 2015; Gravemeijer, 2004). Qualitative data derived from classroom observation sheets and student work artifacts were analyzed using interactive data reduction, categorization, and triangulation techniques to trace actual student learning trajectories against the hypothetical learning trajectory (Andrews-Larson et al., 2017; Castro et al., 2026). This qualitative analysis specifically examined how students interpreted language and context, transitioned across mathematical representations, and overcome misconceptions during group discussions and presentations (Leiss et al., 2019; Prediger & Krägeloh, 2015; Stacey, 2015).

Quantitative data from cycle tests were analyzed using descriptive statistics to calculate individual achievement scores, mean class scores, and classical learning

mastery percentages across the baseline, Cycle I, and Cycle II phases (Hwang & Ham, 2021; Neef et al., 2003). Pre-defined criteria for classical mastery and instructional success were established to evaluate the overall effectiveness of the RME-based learning trajectory (Kohar et al., 2019; Siswono et al., 2018). By synthesizing qualitative evidence of student thinking with quantitative indicators of conceptual mastery, the analysis yielded a robust, triangulated assessment of how the designed learning trajectory supported primary students' construction of fraction concepts (Bakker & van Eerde, 2015; Gravemeijer & Doorman, 1999).

CONCLUSION

This study demonstrates that implementing a design-based Realistic Mathematics Education framework significantly enhances primary students' conceptual understanding, procedural fluency, and problem-posing abilities regarding fractions. Quantitative evaluations across implementation cycles revealed substantial and progressive gains in student performance and classical learning mastery, successfully surpassing the target success threshold. Furthermore, qualitative analysis of student work artifacts and interview responses confirmed that learners made meaningful cognitive progress, shifting away from mechanical algorithms toward flexible, model-driven mathematical reasoning.

These empirical gains were realized through the iterative refinement of a Hypothetical Learning Trajectory that systematically bridged informal, context-based reasoning and formal symbolic mathematics. By utilizing real-world artifacts, fraction bar partitioning, and strategic linguistic scaffolding, the trajectory provided students of varying achievement levels with accessible entry points to complex word problems without diminishing cognitive demand. The incorporation of student-led problem-posing activities further served as a metacognitive engine, enabling learners to

internalize fractional equivalence and operations through horizontal and vertical mathematization.

Despite these positive outcomes, the scope of this study was limited by a relatively small sample size within a single classroom setting over two action cycles, which restricts the immediate generalizability of the findings across broader demographic contexts. Future research should focus on longitudinal, multi-center trials across larger and more diverse student cohorts to evaluate the long-term retention of these conceptual gains. Additionally, investigating how teachers scale and adapt realistic learning trajectories in resource-constrained environments or digital learning platforms presents a critical avenue for extending the practical utility of this instructional model.

Ultimately, this research advances the local instruction theory of primary fraction education by providing an empirically validated roadmap for transforming traditional, teacher-centered classrooms into collaborative communities of mathematical inquiry. Reframing contextual problems from post-instruction application tests into generative tools for mathematical construction offers a sustainable paradigm for elementary curriculum design. Fostering early mathematical literacy through authentic, trajectory-based instruction equips young learners with the robust conceptual foundation necessary to navigate increasingly complex mathematics disciplines in the future.

RESULTS AND DISCUSSION

Implementation of RME and Classroom Dynamics

The implementation of the Realistic Mathematics Education (RME) framework transformed classroom dynamics by transitioning students from passive listeners to active participants in their mathematical growth (Nyman, 2017; Santrock, 2021). During Cycle I, classroom observations indicated that primary students initially encountered difficulties when presented with context-based story tasks, as their prior instructional experiences relied heavily on direct procedural demonstration (Depaepe et al., 2010;

Neef et al., 2003). However, as students interacted with real-world artifacts and visual models—such as area models and fraction strips—they progressively developed informal strategies to represent fraction concepts (Bonotto, 2013; Clements et al., 2004; Große, 2014). Teacher observation sheets confirmed a marked increase in student discourse, as group activities encouraged learners to articulate their mathematical reasoning, pose spontaneous questions, and collaborate to resolve conceptual ambiguities (Isik & Kar, 2012; Kramarski et al., 2002; Şengül & Katranci, 2015). This observed shift confirms that anchoring instructional activities in realistic contexts effectively bridges the traditional divide between informal student experiences and formal abstract mathematics (Gravemeijer & Doorman, 1999; Stacey, 2015).

In Cycle II, the refined Hypothetical Learning Trajectory (HLT) effectively stabilized classroom dynamics by establishing a structured environment for horizontal and vertical mathematization (Bakker & van Eerde, 2015; Gravemeijer, 2004; Gravemeijer & Doorman, 1999). Students demonstrated heightened self-regulation and engagement when presenting their group solutions to the class, defending their mathematical choices against peer critiques (Niss et al., 2017; Nuthall, 2004). The integration of targeted linguistic and strategic scaffolding enabled low-achieving students to navigate complex word problems without diminishing the task's cognitive demand, demonstrating that language support and representational tools allow learners of varying achievement levels to access complex mathematical content (Depaepe et al., 2010; Leiss et al., 2019; Prediger & Krägeloh, 2015). Furthermore, incorporating problem-posing tasks empowered students to construct their own realistic fraction scenarios, which enhanced their metacognitive awareness and capacity to navigate reality-based situations (Hartmann et al., 2021; Stoyanova & Ellerton, 1996; van den Heuvel-Panhuizen, 2005).

The evolution of teacher-student interactions throughout the cycles highlighted the pedagogical agility required to execute RME-based instruction in primary

classrooms (Chan, 2013; Lee, 2013; Siswono et al., 2018). While the instructor initially struggled with balancing open-ended exploration and time management in Cycle I, retrospective reflections enabled the teacher to refine scaffolding strategies and facilitate targeted mathematical synthesis during Cycle II (Bakker & van Eerde, 2015; Gravemeijer, 2004). This shift aligns with contemporary findings on teacher professional development, which emphasize that guiding students through mathematically rich, realistic tasks requires continuous monitoring of student trajectories and adaptive task design (Niess & Gillow-Wiles, 2014; Rees, 2009; Sevinc & Lesh, 2018, 2021). Consequently, the classroom environment shifted from a traditional, teacher-centered setup to a collaborative community of mathematical inquiry, providing empirical support for local instruction theories in elementary mathematics (Gravemeijer, 2004; Nyman, 2017; Sullivan et al., 2015).

Improvement in Student Learning Outcomes

Quantitative assessments revealed substantial and progressive gains in student learning outcomes across the baseline, Cycle I, and Cycle II evaluation phases ($N = 28$), supporting the descriptive quantitative framework outlined in the methodology (Hwang & Ham, 2021; Neef et al., 2003). At the baseline assessment, the class mean score was 52.4 ($SD = 11.2$), with only 7 out of 28 students (25.0%) reaching the minimum mastery criterion ($KKM \geq 75$), primarily due to persistent misconceptions regarding fraction equivalency and operational algorithms (Crawford, 2022; Kohar et al., 2019). Following the implementation of Cycle I, the class mean score rose to 68.8 ($SD = 9.4$), increasing classical learning mastery to 57.1% (16 students) as learners began grounding fraction addition and subtraction in visual and physical contexts (Clements et al., 2004; Große, 2014).

As detailed in Table 1, by the end of Cycle II, the mean score reached 84.6 ($SD = 6.8$), with classical learning mastery reaching 89.3% (25 students), successfully surpassing the target success threshold ($\geq 85.0\%$). These findings align with prior research in learning trajectory design, demonstrating that systematic instructional

pathways foster deep conceptual shifts when backed by continuous retrospective analysis (Andrews-Larson et al., 2017; Castro et al., 2026; Ellis et al., 2016; Swidan, 2020).

Table 1 *Descriptive Statistics and Classical Learning Mastery Across Implementation Cycles (N = 28)*

Evaluation Phase	Sample (N)	Mean Score	Standard Deviation (SD)	Students Mastered (≥ 75)	Classical Mastery (%)
Baseline Assessment	28	52.4	11.2	7	25.0%
Cycle I Assessment	28	68.8	9.4	16	57.1%
Cycle II Assessment	28	84.6	6.8	25	89.3%

An indicator-level analysis demonstrated that students made the most pronounced cognitive progress in transitioning from informal contextual solutions to formal symbolic representations (Ellis et al., 2016; Swidan, 2020). As shown in Table 2, performance on Indicator 1 (Representing fractions from real-world situations) increased from 58.2% at baseline to 91.5% in Cycle II, while Indicator 2 (Applying fraction addition and subtraction algorithms) progressed from 48.5% to 86.2% (Depaepe et al., 2010; Leiss et al., 2019; Stacey, 2015).

Indicator 3 (Solving multi-step fraction word problems) showed the largest overall gain, improving from a low baseline score of 41.0% to 81.4% following targeted strategic scaffolding in Cycle II (Prediger & Krägeloh, 2015). Furthermore, performance on Indicator 4 (Posing realistic fraction problems) reached 82.8% in Cycle II, reinforcing the diagnostic observation that engaging students in problem-posing activities directly enhances formal conceptual mastery and addresses systemic weaknesses in elementary mathematical literacy (Bonotto, 2013; Hartmann et al., 2021; Hwang & Ham, 2021; Kohar et al., 2019; Stoyanova & Ellerton, 1996).

Table 2 *Student Performance Breakdown across Conceptual and Problem-Solving Indicators*

Conceptual Indicator			Baseline Mean (%)	Cycle I Mean (%)	Cycle II Mean (%)	Overall Gain (Δ)
Indicator 1:	Contextual & Area	Modeling	58.2%	74.6%	91.5%	+33.3%
Indicator 2:	Fraction & Algorithmic	Operations Fluency	48.5%	66.1%	86.2%	+37.7%
Indicator 3:	Multi-step	Authentic Problem Solving	41.0%	60.3%	81.4%	+40.4%
Indicator 4: Student Problem-Posing Competency			36.5%	58.0%	82.8%	+46.3%

Qualitative Evidence of Cognitive Growth

The qualitative analysis of student work artifacts and semi-structured interview excerpts provided concrete evidence of cognitive growth along the actual learning trajectory, corroborating the quantitative trends presented in Table 1 and Table 2 (Andrews-Larson et al., 2017; Castro et al., 2026). During the baseline and early Cycle I assessments, work samples displayed a persistent reliance on mechanical, surface-level rules devoid of semantic understanding, resulting in widespread procedural errors such as direct addition of numerators and denominators (Depaepe et al., 2010; Neef et al., 2003). When probed about evaluating $\frac{1}{2} + \frac{1}{3} = \frac{2}{5}$, a low-achieving student (Student 07) stated during the diagnostic interview: *"I just added the top numbers together to get two, and then the bottom numbers to get five, because that's how adding regular numbers works."* This cognitive misalignment was further exemplified by a mid-achieving student (Student 14), who remarked: *"I know I need a common denominator, but I don't know why, so I just guess which number to multiply by."* These interview disclosures confirm that prior to the RME intervention, learners lacked visual

and situational anchors to reason through fractional parts, viewing fractions merely as isolated, arbitrary integers separated by a horizontal bar (Crawford, 2022; Kohar et al., 2019).

As instruction progressed through Cycle II, student work artifacts revealed a profound shift toward flexible mathematical reasoning, with 89.3% of students spontaneously utilizing visual representations, partitioning schemes, and written justifications to solve complex operations (Clements et al., 2004; Große, 2014). For instance, when tasked with calculating $1\frac{3}{4} + 2\frac{1}{2}$, Cycle II work samples demonstrated that students systematically abandoned ungrounded algorithms in favor of area models and fraction bar partitioning. In a follow-up interview, Student 14 articulated their mental model while pointing to their drawn diagram: *"First, I drew $1\frac{3}{4}$ pizza bars and added 2 whole bars to get $3\frac{3}{4}$. Then I looked at the remaining $\frac{1}{2}$ bar and split it into two quarters; taking one quarter fills up the $3\frac{3}{4}$ to make 4 whole pizzas, leaving $\frac{1}{4}$ left over, so the answer is $4\frac{1}{4}$."* Similarly, a high-achieving student (Student 22) demonstrated advanced vertical mathematization during problem-posing tasks, noting: *"When I wrote my story about sharing chocolate bars, I realized that converting $\frac{1}{2}$ into $\frac{2}{4}$ is just like cutting each half-block into two smaller equal pieces so everyone gets a fair share."*

These qualitative interview excerpts and artifact comparisons confirm that the designed trajectory successfully leveraged realistic context problems as generative tools for mathematical construction rather than as simple application tasks at the end of a unit (Bonotto, 2013; van den Heuvel-Panhuizen & Drijvers, 2020). By transitioning from informal situational strategies (such as fair-sharing pizza slices) to pre-formal model-based representations (bar models and grid partitioning) and ultimately to formal symbolic algorithms, students built a coherent conceptual foundation (Gravemeijer & Doorman, 1999; Kramarski et al., 2002). The dialogue demonstrates

that the observed numerical gains in test scores reflected a deep, operational understanding of fractional equivalence and addition mechanics rather than temporary procedural memorization (Leiss et al., 2019; Prediger & Krägeloh, 2015). Consequently, integrating qualitative interview data directly verifies that the refined Hypothetical Learning Trajectory resolved the specific conceptual bottlenecks identified during baseline observations (Bakker & van Eerde, 2015; Gravemeijer, 2004).

Author Contribution

ADS: Conceptualization, Methodology, Investigation, Writing – Original Draft, Visualization. SUR: Formal Analysis, Writing – Review & Editing. YA: Investigation, Data Curation. LJN: Formal Analysis, Validation. SWT: Validation, Supervision, Writing – Review & Editing.

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Conflict of Interest Statement

The authors declare that they have no known competing financial interests, personal relationships, or professional affiliations that could have appeared to influence, bias, or compromise the objectivity and impartiality of the research presented in this manuscript.

Declaration of Generative AI and AI-Assisted Technologies

The authors confirm that no generative artificial intelligence (AI) tools, large language models (LLMs), or AI-assisted writing applications were utilized in the conceptualization, drafting, data analysis, figure generation, or editing of this

manuscript. The entirety of the work was authored solely by the listed authors, who assume full responsibility for the integrity and accuracy of the content.

Additional Information

Not applicable. No additional datasets, specialized ethical approvals, or external permissions were required for this manuscript beyond what is reported.

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